A class of randomly generated semi-infinite programming test problems

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Optimization 2004 - FCUL - Lisbon - Portugal

25-28 July

Outline

- Semi-Infinite Programming (SIP)
- Motivation
- Terminology/Optimality conditions
- Signomials and extended signomials
- Randomly generated constraints
- Objective function of the randomly generated problem
- The algorithm
- Example (NSIPS output) and conclusions

Semi-Infinite Programming (SIP)

$$\begin{aligned} \min_{x \in R^n} \; & f(x) \\ s.t. \;\; & g_i(x,t) \le 0, \; i = 1, ..., m \\ & h_i(x) \le 0, \; i = 1, ..., o \\ & h_i(x) = 0, \; i = o+1, ..., q \\ & \forall t \in T \subset R^p \end{aligned}$$

$f(x)$ is the objective function, $h_i(x)$ are the finite constraint functions, $g_i(x,t)$ are the infinite constraint functions and T is, usually, a cartesian product of intervals ($[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_p, \beta_p]$)

Motivation

For most SIP problems, the exact solutions are not known a priori.

This makes the selection of the best algorithm for SIP a difficult task (NSIPS solver).

The existence of randomly generated SIP test problems (with known solutions) provides a way to evaluate accuracy, efficiency and reliability of known SIP algorithms.

Terminology

For the remaining of the talk, we denote

$$\min_{x \in R^n} \ f(x)$$
$$s.t. \ \ x \in X,$$

with

$$\begin{aligned} X = \{x \in R^n |\ g_u(x,t) \leq 0, & \qquad u = 1,\ldots,m, \forall t \in T, \\ h_v(x) = 0, & \qquad v = 1,\ldots,o, \\ h_v(x) \leq 0, & \qquad v = o+1,\ldots,q\} \end{aligned}$$

as the **upper level problem**.

Terminology - cont.

and

$$\max_{t \in T} \; g_u(x,t), \; u = 1, \dots, m,$$

as the **lower level subproblems**.

Let ς_u be the number of global maxima of the lower level subproblem u, which make the infinite constraint $g_u(x,t) \leq 0$ active.

${}^{u}_{\kappa}t^*$, for $u = 1, \dots, m$ and $\kappa = 1, \dots, \varsigma_u$ are the solutions to lower level subproblems.

Optimality conditions - lower level problem

$$\forall t \in T \equiv [\alpha_1, \beta_1] \times \cdots \times [\alpha_p, \beta_p] \Leftrightarrow \begin{cases} \alpha_j - t_j \leq 0 & j = 1, \ldots, p \\ t_j - \beta_j \leq 0 & j = 1, \ldots, p \end{cases}$$

$(t = (t_1, \ldots, t_p))$

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$(t = (t_1, \ldots, t_p))$

Given $\bar{x}$ (approximation to the upper level problem solution).

The Lagrangian of the lower level subproblem u is

$$\mathcal{L}_u(t, {}^u\gamma^{lb}, {}^u\gamma^{ub}) = g_u(\overline{x}, t) + \sum_{j=1}^{p} {}^u\gamma_j^{lb}(\alpha_j - t_j) + \sum_{j=1}^{p} {}^u\gamma_j^{ub}(t_j - \beta_j),$$

${}^u\gamma^{lb}$, ${}^u\gamma^{ub} \in R^p$ are the Lagrange multipliers vectors.

Lower level first order KKT conditions

The first order KKT conditions for a local maximum:

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complementarity
$$\begin{cases} {}^u\gamma_j^{lb^*}(\alpha_j - {}^u t_j^*) = 0, \;\; j = 1, \ldots, p \\ {}^u\gamma_j^{ub^*}({}^u t_j^* - \beta_j) = 0, \;\; j = 1, \ldots, p \end{cases}$$

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Lagrange multipliers positiveness

$${}^u\gamma_j^{lb^*}, {}^u\gamma_j^{ub^*} \geq 0, \quad j = 1, \ldots, p$$

Lower level second order KKT conditions

The second order sufficient condition:

$$Z^T \nabla^2_{tt} \mathcal{L}_u({}^ut^*, {}^u\gamma^{lb^*}, {}^u\gamma^{ub^*}) Z \prec 0,$$

Z is a basis for the null space of the active constraints Jacobian at ${}^ut^*$.

Upper level Lagrangian

The upper level Lagrangian function is

$$L(x,\lambda,\delta) = f(x) + \sum_{v=1}^{q} \lambda_v h_v(x) + \sum_{u=1}^{m} \sum_{\kappa=1}^{\varsigma_u} {}^u_\kappa\delta g_u(x, {}^u_\kappa t^*),$$

$\lambda = (\lambda_1, \ldots, \lambda_q)^T$ is the Lagrange multipliers vector (finite constraints).

${}^u\delta = ({}^u_1\delta, \ldots, {}^u_{\varsigma_u}\delta)^T$ is the multipliers vector (infinite constraint $g_u(x,t) \leq 0$ $(u = 1, \ldots, m)$).

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$$\left\{ \begin{array}{l} \lambda_v^* \geq 0, \quad v = o+1, \ldots, q \\ {}_\kappa^u\delta^* \geq 0, \quad u = 1, \ldots, m, \quad \kappa = 1, \ldots, \varsigma_u \end{array} \right.$$

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Each ${}^u_\kappa t^*$ satisfies the KKT conditions of the lower level subproblem u.

Upper level second order sufficient condition

The second order sufficient condition for a minimum:

$$\nabla^2_{xx} L(x^*, \lambda^*, \delta^*) \succ 0.$$

Signomial and extended signomials

Signomials are generalized polynomials of the form

$$s(x) = \sum_{\eta=1}^{k} c_\eta \prod_{\zeta=1}^{n} x_\zeta^{a_{\zeta\eta}}, \;\; x > 0,$$

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$$s(x) = \sum_{\eta=1}^{k} c_\eta \prod_{\zeta=1}^{n} x_\zeta^{a_{\zeta\eta}}, \; x > 0,$$

and the extended signomials

$$s^e(x,t) = \left(\sum_{\eta=1}^{k} c_\eta^e \prod_{\zeta=1}^{n} x_\zeta^{a_{\zeta\eta}^e} \right) \prod_{l=1}^{p} \sin^2(t_l b_l \pi), \; x, c_\eta^e, b_l > 0,$$

where c_η, $a_{\zeta\eta}$, c_η^e, $a_{\zeta\eta}^e$ and b_l are real numbers.

Randomly generated constraints

m extended signomials $s_1^e, \ldots, s_m^e$ and $q+1$ signomials $s_0, \ldots, s_q$ are randomly generated.

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$$h_v(x) = s_v(x) - s_v(x^*), \;\; v = 1, \ldots, o + q_a$$

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$m_a(\leq m)$ is the number of infinite active constraints, $q_a(\leq q - o)$ is the number of finite inequality active constraints, ${}^u t^*$ is a global maximum of the extended signomial u and μ_u^e and μ_v are positive randomly generated real numbers.

Objective function

The objective function is defined as

$$f(x) = s_0(x) + \frac{1}{2}x^T\mathbf{H}x + \mathbf{b}^T x + \mathbf{a}$$

where $\mathbf{H} \in R^{n\times n}$, $\mathbf{b} \in R^n$ and $\mathbf{a} \in R$ are given by

$$\mathbf{H} = -\nabla^2 s_0(x^*) - \sum_{v=1}^{o+q_a} \lambda_v^* \nabla^2 h_v(x^*) - \sum_{u=1}^{m_a} \sum_{\kappa=1}^{\varsigma_u} {}_\kappa^u\delta^* \nabla_{xx}^2 g_u(x^*, {}_\kappa^u t^*) + \mathbf{P}$$

$$\mathbf{b} = -\nabla s_0(x^*) - \mathbf{H}x^* - \sum_{v=1}^{o+q_a} \lambda_v^* \nabla h_v(x^*) - \sum_{u=1}^{m_a} \sum_{\kappa=1}^{\varsigma_u} {}_\kappa^u\delta^* \nabla_x g_u(x^*, {}_\kappa^u t^*)$$

$$\mathbf{a} = -s_0(x^*) - \tfrac{1}{2}(x^*)^T \mathbf{H}x^* - \mathbf{b}^T x^*$$

$\mathbf{P} \in R^{n\times n}$ is a positive definite matrix.

Lagrange multipliers

The elements of the multiplier vectors are randomly generated real numbers, satisfying

$$\lambda_v^* > 0, \quad v = o+1, \ldots, o+q_a,$$

$$\lambda_v^* = 0, \quad v = o+q_a+1, \ldots, q,$$

$$^u_\kappa\delta^* > 0, \quad u = 1, \ldots, m_a, \quad \kappa = 1, \ldots, \varsigma_u$$

Note that $\varsigma_u = 0$ for $u = m_a + 1, \ldots, m$.

For practical purposes, we consider $T = [0,1] \times \cdots \times [0,1]$ and $\mathbf{P}$ a diagonal matrix of the form $\mathbf{P} = diag(\rho_i)$, $\rho_i > 0$, $i = 1, \ldots, n$.

The randomly generated lower level subproblems

Given x^*, the lower level subproblem for each $u = 1, \ldots, m_a$ is given by

$$\max_{t \in [0,1]^p} g_u(x^*, t) \equiv s_u^e(x^*, t) - s_u^e(x^*, {}^u t^*).$$

Proposition 1. *Let ${}^u_\kappa t^*$ be defined by*

$$\begin{cases} {}^u_1 t_l^* = 1 & \text{if } {}^u b_l \leq \frac{1}{2}; \\ {}^u_\kappa t_l^* = \frac{1}{2\,{}^u b_l} + \frac{(\kappa - 1)}{{}^u b_l}, \quad \kappa = 1, \ldots, \varsigma_u & \text{if } {}^u b_l > \frac{1}{2}, \end{cases}$$

with $l = 1, \ldots, p$.

Then these points satisfy the first and second order KKT conditions for the lower level subproblem u *at* x^*.

Optimality conditions for the SIP problem

Proposition 2. *Let the SIP problem be randomly generated as previously described. The first and second order KKT conditions of the upper level problem are satisfied for a given* x^* *and Lagrange multipliers.*

The algorithm

1. Input parameters: n, p, m, m_a, o, q_a, q, k, L, Lb and La.

2. Randomly generate the data for the signomials ${}^{v}c_{\eta} \in [-\frac{L}{2}, \frac{L}{2}]$, $v = 0, \ldots, q$, ${}^{u}c^{e}_{\eta} \in]0, \frac{L}{2}]$, $u = 1, \ldots, m$, ${}^{v}a_{\zeta\eta} \in [-\frac{La}{2}, \frac{La}{2}]$, $v = 0, \ldots, q$, $\zeta = 1, \ldots, n$, $\eta = 1, \ldots, k$, ${}^{u}a^{e}_{\zeta\eta} \in [-\frac{La}{2}, \frac{La}{2}]$, $u = 1, \ldots, m$, $\zeta = 1, \ldots, n$, $\eta = 1, \ldots, k$ and ${}^{u}b_{l} \in]0, Lb]$, $u = 1, \ldots, m$, $l = 1, \ldots, p$.

3. Randomly generate the slacks for the inactive infinite constraints ($\mu^{e}_{u} \in]0, L]$, $u = m_a + 1, \ldots, m$) and the slacks for the inactive finite constraints ($\mu_{v} \in]0, L]$, $v = o + q_a + 1, \ldots, q$).

4. Randomly generate the upper level problem solution x^{*}_{l}, $l = 1, \ldots, n$.

The algorithm

5. Compute the first (closest to the left bound of T) lower level subproblems global solution, ${}^u_1t^*_l$, $l = 1, \dots, p$, $u = 1, \dots, m$

$$\begin{cases} {}^u_1t^*_l = 1 & \text{if } {}^ub_l \leq \frac{1}{2}, \\ {}^u_1t^*_l = \frac{1}{2\,{}^ub_l} & \text{otherwise.} \end{cases}$$

6. Compute the number of global maxima of each lower level subproblem,

$$\varsigma_u = \prod_{l=1}^{p} \left[{}^ub_l + 0.5\right]_- , \;\; u = 1, \dots, m_a,$$

where $[y]_-$ is the maximum integer lower or equal to y.

The algorithm

7. Randomly generate the Lagrange multipliers for the upper level problem, λ_i^*, $i = 1, \ldots, o + q_a$ ($\lambda_i^* \in [-\frac{La}{2}, \frac{La}{2}]$, $i = 1, \ldots, o$, $\lambda_i^* \in]0, La]$, $i = o + 1, \ldots, o + q_a$), ${}^u_\kappa\delta^* \in]0, La]$, $u = 1, \ldots, m_a$, $\kappa = 1, \ldots, \varsigma_u$.

8. Define the signomials and extended signomials.

9. Compute the signomials and extended signomials at x^* and ${}^u_1t^*$, $u = 1, \ldots, m$. Compute their derivatives, w.r.t. x, at x^* and ${}^u_1t^*$. (Not shown).

10. Randomly generate the diagonal matrix $\mathbf{P}$ with positive ($]0, L]$) elements.

The algorithm

11. Compute the objective constants $\mathbf{H}$, $\mathbf{b}$ and $\mathbf{a}$.

12. Define the objective function and constraints.

NSIPS discretization method output

```
[aivaz@linux nsips]$ export nsips_options='method=disc_hett'
[aivaz@linux nsips]$ ../ampl ../sipmod/random.mod
Generating problem...option randseed 1070240335;
Done.
NSIPS: Discretization method selected Hettich version
Nx=4     Nt=2
h=(0.010000,0.010000)
Refinements=3
```

NSIPS discretization method output

```
Iter    Grid    Nnlsp   Nacvi   FullGrid
    Nonlinear used 3 iterations Obj=-0.006813 Inform=6
0   10201   40808   2   --
    Nonlinear used 1 iterations Obj=-0.006812 Inform=0
1   40401   12  2   Yes
    Nonlinear used 3 iterations Obj=-0.002451 Inform=0
1   40401   24  3   No
    Nonlinear used 0 iterations Obj=-0.002451 Inform=1
2   361201  16  15  Yes
```

...

NSIPS discretization method output

```
Solution found
Objective= -0.002451
x=(3.646384,1.885763,3.294467,2.650242)

Exact solution
Objective= 0.000000
x^*=(3.646452,1.885912,3.294240,2.650256)
```

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- The file `random.mod` can be changed to tune the randomly generated problem;
- This approach can also be used for multi-local optimization;

The End

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